Chinese Olympiad 2003
IMO Team Selection Contest
Official Solutions

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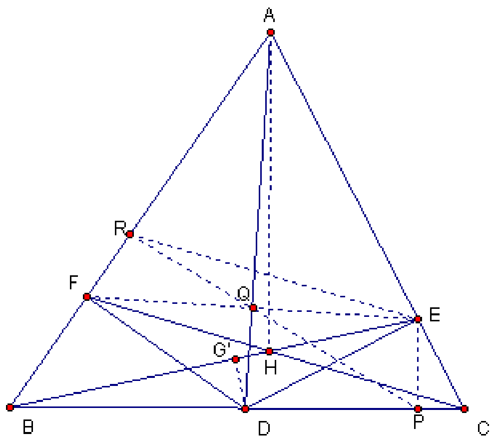
Problem 1 ABC is an acute-angled triangle. AD is the bisector of $\angle A$. Let E, F be the feet of perpendiculars from D to AC, AB respectively. Join BE and CF and suppose they meet at H. The circumcircle of triangle AFH meets BE at G. Prove that the triangle constructed from BG, CE and BF is right-angled.

Proof. Suppose G' is the feet of perpendicular from D to BE. By Pythagoras' Theorem, $BG^2 = G'E^2 = BD^2 - DE^2 = BD^2 - DF^2 = BF^2$. Hence BG', $G'E$ and BF can construct a right-angled triangle. We only to prove that G' is exactly G, and it is sufficient to prove that A, F, G', H lie on the same circle.



Join EF and we find that AD perpendicularly bisects AD. Suppose AD meets EF at Q and let P be the feet of perpendicular from E to BC. The extension of PQ meets AB at R and then we join RE.

Since Q, D, P, E are on the same circle, $\angle QPD = \angle QED$. A, F, D, E are also on the same circle, $\angle QED = \angle FAD$. So A, R, D, P lie on the same circle.

Notice that $\angle RAQ = \angle DAC$ and $\angle ARQ = \angle ADC$, thus, triangle

ARQ is similar to ADC and $\dfrac{AR}{AQ} = \dfrac{AD}{AC}$. Hence $\dfrac{AR}{AF} = \dfrac{AE}{AC}$. Therefore, RE is parallel to FC and $\angle AFC = \angle ARE$.

Because A, R, D, P are on the same circle and so are G', D, P, E, $BG' \cdot BE = BD \cdot BP = BR \cdot BA$, we get that A, R, G', E lie on the same circle. Hence, $\angle AG'E = \angle ARE = \angle AFC$. We finally know that A, F, G', H are on the same circle. □

Problem 2 Suppose $A \subseteq \{0, 1, \ldots, 29\}$. It satisfies that for any integer k and any two members $a, b \in A(a, b$ is allowed to be same), $a + b + 30k$ is always not the product of two consecutive integers. Please find A with largest possible cardinality.

Solution. The answer is $A = \{3l + 2 \mid 0 \leq l \leq 9\}$.

Suppose A satisfy the conditions and $|A|$ is largest. Suppose q is the product of two consecutive integers, thus

$$q \equiv 0, 2, 6, 12, 20, 26 \pmod{30}.$$

For any $a \in A$, we have

$$2a \not\equiv 0, 2, 6, 12, 20, 26 \pmod{30}.$$

which is,

$$2a \not\equiv 0, 1, 3, 6, 10, 13, 15, 16, 18, 21, 25, 28 \pmod{30}.$$

Hence, $A \subseteq \{2, 4, 5, 7, 8, 9, 11, 12, 14, 17, 19, 20, 22, 23, 24, 26, 27, 29\}$. The latter set is the union of the following ten disjoint sets: $\{2, 4\}$, $\{5, 7, \}$, $\{8, 12\}$, $\{9, 11\}$, $\{14, 22\}$, $\{17, 19\}$, $\{20\}$, $\{23, 27\}$, $\{26, 24\}$, $\{29\}$ and every set contains at most one element of A, so $|A| \leq 10$.

If $|A| = 10$, then every set contains exactly at most one element in A, thus $20 \in A$ and $29 \in A$.

From $20 \in A$ we know that $12 \notin A$ and $22 \notin A$, hence $8 \in A$ and $14 \in A$. This implies that $4 \notin A$, $24 \notin A$, therefore, $2 \in A$ and $26 \in A$.

From $29 \in A$ we know that $7 \notin A$ and $27 \notin A$, hence $5 \in A$ and $23 \in A$. This implies that $9 \notin A$, $19 \notin A$, therefore, $11 \in A$ and $17 \in A$.

From all the above, we have $A = \{2, 5, 8, 11, 14, 17, 20, 23, 26, 29\}$, and it does satisfy the conditions. □

Problem 3 Suppose $A \subset \{(a_1, a_2, \dots, a_n) \mid a_i \in \mathbb{R}, i = 1, 2 \dots, n\}$. For any $\alpha = (a_1, a_2, \dots, a_n) \in A$ and $\beta = (b_1, b_2, \dots, b_n) \in A$, we define

$$\gamma(\alpha, \beta) = (|a_1 - b_1|, |a_2 - b_2|, \dots, |a_n - b_n|),$$

$$D(A) = \{\gamma(\alpha, \beta) \mid \alpha, \beta \in A\}.$$

Please show that $|D(A)| \geq |A|$.

Proof. We use mathematical induction for n and the cardinality of A.

If A contains only one element, $D(A)$ contains a zero vector. The conclusion holds.

If $n = 1$, suppose $A = \{a_1 < a_2 < \cdots < a_m\}$, thus

$$\{0, a_2 - a_1, a_3 - a_1, \dots, a_m - a_1\} \subseteq D(A).$$

Hence $|D(A)| \geq |A|$.

Assume $|A| > 1$ and $n = 1$. Define $B = \{(x_1, x_2, \dots, x_{n-1})|$ there exists x_n such that $(x_1, x_2, \dots, x_{n-1}, x_n) \in A\}$. By the induction hypothesis we have $|D(B)| \geq |B|$.

For every $b \in B$, let $A_b = \{x_n \mid (b, x_n) \in A\}$, $a_b = \max\{x | x \in A_b\}$, $C = A \setminus \{(b, a_b) | b \in B\}$. Thus, $|C| = |A| - |B|$.

Since $|C| < |A|$, by the induction hypothesis $|D(C)| \geq |C|$.

On the other hand, $D(A) = \bigcup_{D \in D(B)} \{(D, |a - a'|) \mid d(b, b') = D$ and $a \in A_b, a' \in A_{b'}\}$.

Similarly, let $C_b = A_b \setminus \{a_b\}$, we have $D(C) = \bigcup_{D \in D(B)} \{(D, |c - c'|) \mid d(b, b') = D$ and $c \in C_b, c' \in C_{b'}\}$.

Notice that for every pair $b, b' \in B$, the maximum difference $|a-a'|(a \in A_b, a' \in A_{b'})$ must be $a = a_b$ or $a' = a_{b'}$. Therefore this maximum difference does not appear in $\{|c-c'| \mid c \in C_b, c' \in C_{b'}\}$.

So for any $D \in D(B)$, the set

$$\{|c-c'| \mid d(b,b') = D \text{ and } c \in C_b, c' \in C_{b'}\}$$

does not contain the maximum value in the set

$$\{|a-a'| \mid d(b,b') = D \text{ and } a \in A_b, a' \in A_{b'}\}.$$

The former set is a proper subset of the latter one.

Now we have

$$\begin{aligned} |D(C)| &\leq \sum_{D \in D(B)} (|\{(D, |a-a'|) \mid d(b,b') = D \text{ and } a \in A_b, a' \in A_{b'}\}| - 1) \\ &\leq |D(A)| - |D(B)|. \end{aligned}$$

Therefore $|D(A)| \geq |D(B)| + |D(C)| \geq |B| + |C| = |A|$. □

Problem 4 Find all functions $f : \mathbb{Z}^+ \to \mathbb{R}$, which satisfies

(a) $f(n+1) \geq f(n)$ for all $n \geq 1$;

(b) $f(mn) = f(m)f(n)$ for all $(m,n) = 1$.

Solution. Obviously $f = 0$ is a solution.

Now assume $f \neq 0$, thus $f(1) \neq 0$. If not, for any $n \in \mathbb{Z}^+$, it holds that $f(n) = f(1)f(n) = 0$, we meet a contradiction here. So $f(1) = 1$.

From (a) we know that $f(2) \geq 1$. Now we discuss in two cases.

Case i. $f(2) = 1$. We may prove that

$$f(n) = 1 \tag{1}$$

for all n. In fact, we know from (b) that $f(6) = f(2)f(3) = f(3)$. Denote $f(3) = a$, $a \geq 1$. Since $f(3) = f(6) = a$, we get from (a) that

$f(4) = f(5) = a$. Use (b) again, we find that for any odd integer p, it holds $f(2p) = f(2)f(p) = f(p)$. From this and (a), we may prove that

$$f(n) = a \tag{2}$$

for all $n \geq 3$. In fact, $a = f(3) = f(6) = f(5) = f(10) = f(9) = f(18) = f(17) = f(34) = f(33) = \cdots$

By (2) and (b), we know that $a = 1$, i.e., $f = 1$. Hence (1) is correct.

Case ii. $f(2) > 1$. Suppose $f(2) = 2^a$, where $a > 0$.

Let $g(x) = f^{1/n}(x)$, thus $g(x)$ satisfies (a) and (b), and $g(1) = 1, g(2) = 2$.

Suppose $k \geq 2$. We get from (a) that

$$\begin{aligned} 2g(2^{k-1} - 1) &= g(2)g(2^{k-1} - 1) = g(2^k - 2) \\ &\leq g(2^k) \leq g(2^k + 2) = g(2)g(2^{k-1} + 1) \\ &= 2g(2^{k-1} + 1); \end{aligned}$$

If $k \geq 3$,

$$\begin{aligned} 2^2 g(2^{k-2} - 1) &= 2g(2^{k-1} - 2) \\ &\leq g(2^k) \leq 2g(2^{k-1} + 2) = 2^2 g(2^{k-2} + 1); \end{aligned}$$

In this way, we may induction to prove that

$$2^{k-1} \leq g(2^k) \leq 2^{k-1} g(3) \tag{3}$$

for all $k \geq 2$. Similarly,

$$g^{k-1}(m)g(m-1) \leq g(m^k) \leq g^{k-1}(m)g(m+1) \tag{4}$$

for all $k \geq 2$ and $m \geq 3$. It is easy to check that (3) and (4) also hold when $k = 1$.

Take $m \geq 3$ and $k \geq 1$, we have $s \geq 1$ such that $2^s \leq m^k < 2^{s+1}$, hence

$$s \leq k \log_2 m < s + 1,$$

which is,

$$k \log_2 m - 1 < s \leq \log_2 m, \tag{5}$$

By (a), we know that $g(2^s) \le g(m^k) \le g(2^{s+1})$, then by (3) and (4),

$$\begin{cases} 2^{s-1} \le g^{k-1}(m)g(m+1) \\ g^{k-1}(m)g(m-1) \le 2^{s-1}g(3). \end{cases}$$

i.e.,

$$\frac{2^{s-1}}{g(m+1)} \le g^{k-1}(m) \le \frac{2^{s-1}g(3)}{g(m-1)}.$$

Therefore,

$$\frac{g(m)}{g(m+1)}2^{s-1} \le g^k(m) \le \frac{g(m)g(3)}{g(m-1)}2^{s-1}.$$

We get from (5) that

$$\frac{g(m)}{4g(m+1)}2^{k\log_2 m} \le g^k(m) \le \frac{g(m)g(3)}{2g(m-1)}2^{k\log_2 m}.$$

so

$$\sqrt[k]{\frac{g(m)}{4g(m+1)}}2^{\log_2 m} \le g(m) \le \sqrt[k]{\frac{g(m)g(3)}{2g(m-1)}}2^{\log_2 m}.$$

Let $k \to \infty$, we get $g(n) = m$ which implies that $f(m) = m^k$.

From all the above, we finally know that $f = 0$ or $f(n) = n^a (a \ge 0)$. $\square$

Problem 5 Suppose $A = \{1, 2, \ldots, 2002\}$ and $M = \{1001, 2003, 3005\}$. B is an non-empty subset of A. B is called a M-free set if the sum of any two numbers in B does not belong to M. If $A = A_1 \cup A_2$, $A_1 \cap A_2 = \emptyset$ and A_1, A_2 are M-free sets, we call the ordered pair (A_1, A_2) a M-partition of A. Find the number of M-partitions of A.

Solution. We call m and n $(m, n \in A)$ are relative if $m + n = 1001$ or 2003 or 3005.

It is clear that the numbers relative to 1 are only 1000 and 2002, and the numbers relative to 2002 are 1 and 1003, and relative to 1003 are 1000 and 2002.

Hence, 1, 1003, 1000, 2002 must be divided into two groups: $\{1, 1003\}$ and $\{1000, 2002\}$. Similarly, we can get other divisions

$$\begin{array}{rcl} \{2, 1004\} & , & \{999, 2001\}; \\ \{3, 1005\} & , & \{998, 2000\}; \\ & \cdots & \\ \{500, 1502\} & , & \{501, 1503\}; \\ \{1001\} & , & \{1002\}. \end{array}$$

Now the 2002 numbers in A have been divided into 501 pairs, 1002 groups in all.

Because every number is only relative to the its correspond group in the same pair, we know that if a group is in A_1, then its correspond group must be in A_2. Therefore the number of M-partitions of A is 2^{501}. □

Problem 6 x_n is a real sequence satisfying $x_0 = 0, x_2 = \sqrt[3]{2}x_1$, x_3 is a positive integers and $x_{n+1} = \frac{1}{\sqrt[3]{4}}x_n + \sqrt[3]{4}x_{n-1} + \frac{1}{2}x_{n-2}$ for $n \geq 2$. How many integers at least belong to this sequence?

Solution. Assume $n \geq 2$,

$$\begin{aligned} & x_{n+1} - \sqrt[3]{2}x_n - \frac{1}{\sqrt[3]{2}}x_{n-1} \\ = \; & \frac{1}{\sqrt[3]{4}}x_n - \sqrt[3]{2}x_n + \sqrt[3]{4}x_{n-1} - \frac{1}{\sqrt[3]{2}}x_{n-1} + \frac{1}{2}x_{n-2} \\ = \; & -\frac{\sqrt[3]{2}}{2}x_n + \frac{\sqrt[3]{4}}{2}x_{n-1} + \frac{1}{2}x_{n-2} \\ = \; & -\frac{\sqrt[3]{2}}{2}\left(x_n - \sqrt[3]{2}x_{n-1} - \frac{1}{\sqrt[3]{2}}x_{n-2}\right). \end{aligned}$$

Since $x_2 - \sqrt[3]{2}x_1 - \frac{1}{\sqrt[3]{2}}x_0 = 0$,

$$x_{n+1} = \sqrt[3]{2}x_n + \frac{1}{\sqrt[3]{2}}x_{n-1} \quad (\forall\, n \geq 1) \tag{1}$$

The characteristic equation of (1) is $\lambda^2 = \sqrt[3]{2}\lambda + \frac{1}{\sqrt[3]{2}}$, from which we get $\lambda = \frac{\sqrt[3]{2}}{2}(1 \pm \sqrt{3})$. And by $x_0 = 0$, we have

$$x_n = A\left(\frac{\sqrt[3]{2}}{2}\right)^n ((1+\sqrt{3})^n - (1-\sqrt{3})^n).$$

So $x_3 = \frac{A}{4}\left(\frac{\sqrt[3]{2}}{2}\right)^3 ((1+\sqrt{3})^n - (1-\sqrt{3})^3) = 3\sqrt{3}A$, from which we get $A = \frac{x_3}{3\sqrt{3}}$. Therefore,

$$x_n = \frac{x_3}{3\sqrt{3}}\left(\frac{\sqrt[3]{2}}{2}\right)^n ((1+\sqrt{3})^n - (1-\sqrt{3})^n). \tag{2}$$

Denote $a_n = \frac{1}{\sqrt{3}}((1+\sqrt{3})^n - (1-\sqrt{3})^n)$, it is obvious that a_n is an even number sequence. From x_3 is a positive integer and (2) we know the necessary condition for a_n being an integer is $3|n$.

$$\begin{aligned} a_{3k} &= \frac{3}{3\sqrt{3}}((1+\sqrt{3})^{3k} - (1-\sqrt{3})^{3k}) \\ &= \frac{3}{3\sqrt{3}}((10+6\sqrt{3})^k - (10-6\sqrt{3})^k) \end{aligned}$$

We know that $3|a_{3k}$.

Let $b_n = (1+\sqrt{3})^n + (1-\sqrt{3})^n, n \geq 0$, b_n is also an even number sequence. It is easy to see that for any non-negative integers m, n, it holds

$$\begin{cases} a_{n+m} = \frac{1}{2}(a_n b_m + a_m b_n), \\ b_{n+m} = \frac{1}{2}(b_n b_m + 3a_n a_m). \end{cases} \tag{3}$$

Let $m = n$ in (3),

$$\begin{cases} a_{2n} = a_n b_n, \\ b_{2n} = \frac{1}{2}(b_n^2 + 3a_n^2). \end{cases} \tag{4}$$

Suppose $a_n = 2^{k_n} p_n, b_n = 2^{l_n} q_n$, where n, k_n, l_n are positive integers and p_n, q_n are odd integers.

Since $a_1 = b_1 = 2$, i.e., $k_1 = l_1 = 1$, we know from (4) that
$k_2 = 2$, $l_2 = 3$,
$k_4 = 5$, $l_4 = 3$, We get by induction that
$k_8 = 8$, $l_8 = 5$.

$$k_{2^m} = \begin{cases} 1, & m = 0, \\ 2, & m = 1, \\ 2^{m-1} + m + 1, & m \geq 2, \end{cases}$$

and

$$l_{2^m} = \begin{cases} 1, & m = 0, \\ 3, & m = 1, \\ 2^{m-1} + 1, & m \geq 2, \end{cases}$$

$\forall\, m_1 > m_2 \geq 2$, we have by (3) that

$$\begin{cases} a_{2^{m_1}+2^{m_2}} = \dfrac{1}{2}(a_{2^{m_1}} b_{2^{m_2}} + a_{2^{m_2}} b_{2^{m_1}}) \\ b_{2^{m_1}+2^{m_2}} = \dfrac{1}{2}(b_{2^{m_1}} b_{2^{m_2}} + 3a_{2^{m_1}} a_{2^{m_1}}) \end{cases}$$

And it implies that

$$\begin{cases} k_{2^{m_1}+2^{m_2}} &=& 2^{m_1-1} + 2^{m_2-1} + m_2 + 1 \\ l_{2^{m_1}+2^{m_2}} &=& 2^{m_1-1} + 2^{m_2-1} + 1 \end{cases}$$

Using induction, we may prove that for $m_1 > m_2 > \cdots > m_r \geq 2$,

$$\begin{cases} k_{2^{m_1}+2^{m_2}+\cdots+2^{m_r}} &=& 2^{m_1-1} + 2^{m_2-1} + \cdots + 2^{m_r-1} + m_r + 1 \\ l_{2^{m_1}+2^{m_2}+\cdots+2^{m_r}} &=& 2^{m_1-1} + 2^{m_2-1} + \cdots + 2^{m_r-1} + 1 \end{cases}$$

That is, when $n = 2^r p$, where $r (r \geq 2)$ is an integer and p is odd, it holds that

$$\begin{cases} k_n &=& \dfrac{n}{2} + r + 1, \\ l_n &=& \dfrac{n}{2} + 1. \end{cases} \tag{5}$$

When $n = 4m + 1$, we have from (3) that

$$a_{4m+1} = \frac{1}{2}(a_{4m} b_1 + a_1 b_{4m}) = a_{4m} + b_{4m}$$

and from (5) that $k_{4m+1} = 2m + 1$.

Similarly, from

$$a_{4m+2} = \frac{1}{2}(a_{4m}b_2 + a_2 b_{4m}) = 2(2a_{4m} + b_{4m})$$

and

$$a_{4m+3} = \frac{1}{2}(a_{4m}b_3 + a_3 b_{4m}) = 2(5a_{4m} + 3b_{4m}),$$

it holds that $k_{4m+2} = k_{4m+3} = 2m + 2$.

From all the above, we get

$$k_n = \begin{cases} \frac{n}{2} + \frac{1}{2}, & n \text{ is odd}, \\ \frac{n}{2} + 1, & n \equiv 2 \pmod 4, \\ \frac{n}{2} + r + 1, & n = 2^r p, r \geq 2, p \text{ is odd}. \end{cases}$$

When $3|n$, we get from (2) that

$$x_n = \frac{x_3}{3} 2^{-\frac{2}{3}n} a_n = \frac{x_3}{3} 2^{k_n - \frac{2}{3}n} p_n,$$

where $3|p_n$. Since $k_3 = 2 = \frac{2}{3} \times 3$, $k_6 = 4 = \frac{2}{3} \times 6$, $k_{12} = 9 > \frac{2}{3} \times 12$ and $k_{24} = 16 = \frac{2}{3} \times 24$, x_3, x_6, x_{12} and x_{24} are all integers.

If $n \not\equiv 0 \pmod 4$, then $k_n \leq \frac{n}{2} + 1$. So

$$k_n - \frac{2}{3}n \leq 1 - \frac{n}{6} < 0 (\forall\ n > 6). \tag{6}$$

If $n \equiv 0 \pmod 4$, since $3|n$, $n = 2^r 3^k q$, where $r \geq 2$, $k \geq 1$ and q does not have a factor 3. We know from (5) that $k_n = 2^{r-1}3^k q + r + 1$. Hence

$$\begin{aligned} k_n - \frac{2}{3}n &= 2^{r-1}3^k q + r + 1 - 2^{r+1}3^{k-1}q \\ &= r + 1 - 2^{r-1}3^{k-1}q \leq r + 1 - 2^{r+1}. \end{aligned}$$

The equality above holds if and only if $k = q = 1$.

When $r > 3$, $2^{r-1} > r + 1$. We know from this that when $r > 3$ or $2 \leq r \leq 3$ and $k \neq 1$ or $q \neq 1$, it holds that

$$k_n - \frac{2}{3}n < 0 \tag{7}$$

From (6) and (7), we get that only $x_0, x_3, x_6, x_{12}, x_{24}$ are integers in the sequence $\{x_n\}$. We finally obtain that at least five numbers in the sequence are integers. □